A NOTE ON EVENT ORDERING SEMANTICS AND THE NOETHERIAN CONSTRAINT

Abstract. Wesley Holliday and Thomas Icard (2013) have recently presented a world-ordering semantics for natural language probability talk and proved that it yields essentially the same logic as a standard probabilistic semantics. But ability of their world ordering semantics to yield the appropriate logic depends on their imposing the constraint the world orders are Noetherian pre-orders. Adam Marushak (2020) has recently objected to this constraint on the grounds that it is (i) not well motivated, and (ii) faces problems involving infinitely large domains. In this note I show how the requirement that the relevant pre-orders be Noetherian can be motivated from the perspective of an event ordering semantics for natural language probability talk.

1. Introduction

Wesley Holliday and Thomas Icard (2013) have recently presented a world-ordering semantics for natural language probability talk and proved that it yields essentially the same logic as a standard probabilistic semantics. But ability of their world ordering semantics to yield the appropriate logic depends on their imposing the constraint the world orders are Noetherian pre-orders. Adam Marushak (2020) has recently objected to this constraint on the grounds that it is (i) not well motivated, and (ii) faces problems involving infinitely large domains. In this note I show how the requirement that the relevant pre-orders be Noetherian can be motivated from the perspective of an event ordering semantics for natural language probability talk.

2. Event Ordering Semantics for Epistemic Comparatives

Definition 2.1. *For any set* Φ *of* atomic sentences*, the set of* $\mathcal{L}$-formulas *is defined to be the smallest set* $\mathcal{L}(\Phi)$ *containing* Φ *that includes* $\top, \bot, \neg\varphi, (\varphi \wedge \psi), \Diamond\varphi, (\varphi \geqslant \psi)$ *whenever it includes* φ *and* ψ.

Other operators will be treated as informal abbreviations in the standard way. I n addition $(\varphi > \psi)$ will abbreviate $(\varphi \geqslant \psi) \wedge \neg(\psi \geqslant \varphi)$ and $\Delta\varphi$ will abbreviate $(\varphi > \neg\varphi)$. Informally $\varphi \geqslant \psi$ is taken to mean 'φ is at least as likely as ψ' and $\Delta\varphi$ meaning 'φ is likely'.

The *event-ordering semantics* for this language is quite simple: formulas are assigned "events," or what would probably more preferably described as propositions, which are modeled as sets of possible worlds, and $\geqslant$ is interpreted as a binary relation on these propositions. In particular, we start with the notion of a *frame*:

Definition 2.2. *A* frame *is a pair $\langle W, \leq \rangle$ such that W is a non-empty set and $\leq$ is a binary relation on $\mathcal{P}(W)$.*

These frames are of course quite simple. One can basically guarantee the logic one wants for the operators $\geqslant$ by stipulating the corresponding condition to hold on the frame. This might be seen as the main objection to them. In the literature on formal semantics, there is a strong preference for views on which the structure of propositions arises from some underlying structure on the space of worlds. In this case, there is a preference for a view on which the relation $\leq$ arises in some way from an underlying relation on possible worlds.

But it is worth noting that by working with a frame $(W, \leq)$ we are not actually *assuming* that the relation $\leq$ does *not* arise from some such relation on W. For we can define subclasses of frames on which $\leq$ is definable from some ordering on worlds. Below we will consider a couple of natural classes of such frames.

One thing to note here is the reversal of the direction of the relation $\leq$ vis-a-vis the operator $\geqslant$. The idea of using $\leq$ as a semantic value in a model for $\geqslant$ is initially a bit awkward because one suggests "greater than or equal" and the other suggests "less than or equal.".But I think eventually when we start looking at some specific kinds of frames this reversal actually makes a bit more sense. Apologies if the reader doesn't share that phenomenology!

We evaluate formulas on frames by equipping frames with *valuations*:

Definition 2.3. *A* **model** *is a tuple* $\mathcal{M} = \langle W, \leq, V \rangle$ *such that* $\mathcal{F} = \langle W, \leq \rangle$ *is a frame and* $V : \Phi \to \mathcal{P}(W)$ *is a function.*

Definition 2.4. *Let* $\mathcal{M} = \langle W, \leq, V \rangle$ *be a model. Then* V *uniquely extends to a map*

$$[\![\cdot]\!]^{\mathcal{M}} : \mathcal{L}(\Phi) \to \mathcal{P}(W)$$

as follows:

$$[\![p]\!]^{\mathcal{M}} = V(p);$$
$$[\![\neg\varphi]\!]^{\mathcal{M}} = W \setminus [\![\varphi]\!]^{\mathcal{M}}$$
$$[\![(\varphi \wedge \psi)]\!]^{\mathcal{M}} = [\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}};$$
$$[\![\Diamond\varphi]\!] = \{w \in W \mid [\![\varphi]\!]^{\mathcal{M}} \neq \emptyset\}$$
$$[\![(\varphi \geqslant \psi)]\!] = \{w \in W \mid [\![\psi]\!]^{\mathcal{M}} \leq [\![\varphi]\!]^{\mathcal{M}}\}$$

We say that φ *is* **valid** *in* $\mathcal{M}$ *if* $[\![\varphi]\!]^{\mathcal{M}} = W$. *We say that* φ *is* **valid** *on a frame* $\mathcal{F}$ *if it is valid on every model based on that frame. We say that* φ *is valid on a class* K *of frames if and only if* φ *is valid on every frame in that class. The* **logic** *of a class of frames is the set of all formulas that are valid on that class.*

The logic of the class of all frames is of course extremely uninteresting. The modal operator $\Diamond$ satisfies the principles of $\mathsf{S5}$ and the operator $\geqslant$ does not satisfy any interesting constraints. But we can find more interesting logics by investigatings frames in which the ordering $\leq$ arises in a natural way from an underlying ordering on worlds.

3. Standard Frames and Injective Frames

Definition 3.1 (Standard Frames)**.** *let* $\mathcal{F} = \langle W, \leq \rangle$ *be a frame and let* $\preceq$ *be a pre-order on* W. *We say that* $\mathcal{F}$ *is* **generated** *by* $\preceq$ *if the following holds*

$$X \leq Y \text{ if and only if there is a function } f : X \to Y \text{ such that } w \preceq f(w)$$

A frame is set to be **standard** *if it is generated by a pre-order.*

The class of standard frames are essentially the event ordering frames which correspond to Kratzer's world ordering models.

Proposition 3.2. *If a frame is standard then it is generated by a unique pre-order.*

Proof. Let $\mathcal{F} = \langle W, \leq \rangle$ be a standard frame. By definition it is generated by some pre-order $\preceq$ on W. Suppose that it is also generated by $\preceq'$. Let $w \preceq v$. Then the unique function $f : \{w\} \to \{v\}$ is such that $w \preceq f(w)$ since $f(w) = v$. So $\{w\} \leq \{v\}$. Since $\preceq'$ generates $\mathcal{F}$, there is a non-decreasing function $g : \{w\} \to \{v\}$. Since $w \mapsto v$ is the unique such function, $w \preceq' v$. $\square$

Since a given standard frame $\langle W, \leq \rangle$ determines the pre-order $\preceq$ from which it is generated, we will sometimes write $\langle W, \preceq \rangle$ for this standard frame when it is natural to do so.

As has been noted, the logic of the class of standard frames is not plausibly the logic of comparative likelihood because of the *disjunction problem*:

Proposition 3.3. $((\varphi \geqslant \psi) \wedge (\varphi \geqslant \chi)) \to (\varphi \geqslant (\psi \vee \chi))$ *is valid on all standard frames.*

Since $((\varphi \geqslant \psi) \wedge (\varphi \geqslant \chi)) \to (\varphi \geqslant (\psi \vee \chi))$ is viewed by many as obviously invalid, most will regard standard frames as poor candidates for providing the logic of comparative likelihood.

To avoid this problem, Holliday and Icard (2013) propose a slightly different class of models that better captures comparative likelihood that uses a notion of "m-lifting." In the event ordering context we can define the desired class of models by slightly tweaking the definition of standard models.

Definition 3.4. *Let $\mathcal{F} = \langle W, \leq \rangle$ be a frame and $\preceq$ a pre-order on W. We say that $\mathcal{F}$ is* **injectively generated** *by $\preceq$ if the following holds:*

$$X \leq Y \textit{ if and only if there is an injection } f : X \hookrightarrow Y \textit{ such that } w \preceq f(w)$$

We say that a frame is **injective** *if it is injectively generated by a pre-order.*

As before if a frame is injective, it is injectively generated by a *unique* pre-order and so we can write $\langle W, \preceq\rangle$ for the frame $\langle W, \leq\rangle$ injectively generated by $\preceq$. Which frame we mean by $\langle W, \preceq\rangle$ will always be clear from context.

The class of injective frames fares at least initially better than the class of standard frames since unlike the latter, the former does not face the disjunction problem:

Proposition 3.5. $((\varphi \geqslant \psi) \wedge (\varphi \geqslant \chi)) \rightarrow (\varphi \geqslant (\psi \vee \chi))$ *is not valid on the class of injective frames.*

But as Marushak (2020) has recently argued, the class of injective frames is also not completely satisfactory. The problem is not that it includes as a theorem a formula that is intuitively not valid. Rather, the logic of injective frames *fails* to include some formulas that intuitively *are* valid. Consider

$$\mathcal{T}(\varphi, \psi) := ((\varphi \geqslant \psi) \rightarrow (\Delta\psi \rightarrow \Delta\varphi))$$

Intuitively this states that if φ is at least as likely as ψ and ψ is likely, then φ is likely. Many will regard this schema as a core part of the logic of comparative likelihood.

Proposition 3.6. *There is an injective frame on which $\mathcal{T}(p, p)$ is not valid.*

Proof. We will basically reproduce the proof of Marushak (2020) Let $\mathcal{F} = \langle \mathbb{N}, \mathbb{N} \times \mathbb{N}\rangle$ be an injective frame. Note that in this frame any function $f : X \rightarrow Y$ between subsets is nondecreasing. Let $\mathcal{M} = \langle \mathbb{N}, \mathbb{N} \times \mathbb{N}, V\rangle$ be a model based on that frame in which $V(p) = \{n \in \mathbb{N} \mid n \text{ is even}\}$. Then the map $n \mapsto 2n$ is a non-decreasing injection from $[\![p \vee \neg p]\!]^{\mathcal{M}}$ to $[\![p]\!]^{\mathcal{M}}$. So $[\![p \vee \neg p]\!]^{\mathcal{M}} \leq [\![p]\!]^{\mathcal{M}}$. Trivially there is a non-decreasing injection $\emptyset \hookrightarrow [\![p \vee \neg p]\!]^{\mathcal{M}}$ but not vice versa. So $[\![\neg(p \vee \neg p)]\!]^{\mathcal{M}} < [\![p \vee \neg p]\!]^{\mathcal{M}}$. But since $[\![\neg p]\!]^{\mathcal{M}}$ is the set of all odd numbers, there is a non-decreasing injection from $[\![p]\!]^{\mathcal{M}}$ to $[\![\neg p]\!]^{\mathcal{M}}$ and vice versa. Thus we see that in $\mathcal{M}$, $[\![\mathcal{T}(p, p)]\!]^{\mathcal{M}} = \emptyset$. (It would suffice to show that it is not identical to W. But as it happens in this case every world is an element of $[\![(p \geqslant (p \vee \neg p)) \wedge \Delta(p \vee \neg p))]\!]^{\mathcal{M}}$ but no world is an element of $[\![\Delta p]\!]^{\mathcal{M}}$.) □

To avoid this problem, Holliday and Icard (2013) propose one further constraint that the underlying relation on injectively bounded frames ought to satisfy that is *not* satisfied in the above model. The constraint is the following:

Definition 3.7 (Noetherian frame)**.** *Let* $\mathcal{F} = \langle W, \preceq \rangle$ *be an injective frame. Then* $\mathcal{F}$ *is* Noetherian *if and only if there does not exist an infinite sequence of distinct worlds:*

$$w_1 \preceq w_2 \preceq w_3 \ldots$$

Obviously the above countermodel is not itself Noetherian. And in fact the logic of the class of Noetherian frames *does* include every instance of $\mathcal{T}(\varphi, \psi)$. Indeed the logic of the class of Noetherian frames can be shown to be the logic of a class of plausible *probabilistic* models for the language under consideration. Some have argued that because of this, there are no logical grounds for preferring a probabilistic semantics for talk of comparative likelihood over a semantics given in terms of orderings of worlds.

In the remaining part of this note I discuss a slightly different route to the Noetherian constraint that *may* provide some motivation for it. The basic idea is to start from some plausible principle concerning the relation $\leq$ of comparative likelihood on *propositions* and use that to motivate the idea that the injective frames in question ought to satisfy the Noetherian constraint.

4. Injectitvely Bounded Frames

The countermodel exhibited above shows that $\mathcal{T}(\varphi, \psi)$ is not part of the logic of injective frames. This countermodel had another odd feature. It invalidated the following schema:

$$\mathcal{R}(\varphi) := (\varphi \geq \top) \to \Box\varphi$$

In the above model we had $[\![p \vee \neg p]\!]^{\mathcal{M}} \leq [\![p]\!]^{\mathcal{M}}$ without its being the case that $[\![p]\!]^{\mathcal{M}} = W$. This feature of the model does not merely arise from our simplification of excluding an accessibility relation in the interpretation of $\Diamond$.If the logic of $\Diamond$ is $\mathsf{S5}$ we could give essentially the same countermodel with an accessibility relation R being the universal relation on W. But $\mathcal{R}(\varphi)$ strikes me as an extremely plausible candidate for being a theorem of the joint logic of epistemic necessity and comparative likelihood. After all, $\Box\top$ is a theorem. So if $\top \leq \varphi$, φ is at least as likely as something that must be true. So how could it be that φ might be false?

This suggests that we are going to need *some* kind of restriction on the class of injective models. A natural thought is to simply impose the constraint by brute force:

Definition 4.1 (Injectively Bounded Frames)**.** *Let $\mathcal{F} = \langle W, \leq \rangle$ be a frame. Then $\mathcal{F}$ is* bounded *if W is the unique element of $\mathcal{P}(W)$ such that $X \leq W$, for all $X \in \mathcal{P}(W)$. $\mathcal{F}$ is* injectively bounded *if and only if it both injective and bounded.*

The basic idea behind these frames is clear enough. We've seen that the injective frames in general predict that epistemic necessity and comparative likelihood come apart. But supposing we want to maintain that a proposition X is as likely as the tautologous proposition W only if X is epistemically necessary, we should then require that $X = W$ whenever $X \leq W$. And once we do impose this constraint, we can see that $\mathcal{R}(\varphi)$ is valid on on the resulting frame. To see this first let's show the following lemma.

Lemma 4.2. *If $\mathcal{F} = \langle W, \leq \rangle$ is injectively bounded and $W \leq X$ then $X = W$.*

Proof. Let $\mathcal{F} = \langle W, \leq \rangle$ be injectively bounded and suppose that $W \leq X$. Let $Y \subseteq W$ be arbitrary. Then $Y \leq W$ by the assumption that $\mathcal{F}$ is injectively bounded. Thus there are non-decreasing injections $f : Y \to W$ and $g : W \to X$. Then $g \circ f : Y \to X$ is a non-decreasing injection and so $Y \leq X$. Since Y was arbitrary, it follows that $X = W$ by the assumption that $\mathcal{F}$ is bounded. $\Box$

We can now observe the validity of $\mathcal{R}(\varphi)$ on injectively bounded frames as follows.

Proposition 4.3. *If $\mathcal{F}$ is injectively bounded , then $\mathcal{R}(\varphi)$ is valid on $\mathcal{F}$.*

Proof. Let $\mathcal{M}$ be an arbitrary model based on $\mathcal{F}$ and suppose that $w \in [\![\varphi \geqslant \top]\!]^{\mathcal{M}}$. It follows then that $[\![\top]\!]^{\mathcal{M}} \leq [\![\varphi]\!]^{\mathcal{M}}$. Since $[\![\top]\!]^{\mathcal{M}} = W$, and $\leq$ is transitive in any injective frame, it follows then that $[\![\varphi]\!]^{\mathcal{M}} = W$, since $\mathcal{F}$ is bounded. Thus $[\![\Box\varphi]\!]^{\mathcal{M}} = W$ and so $w \in [\![\Box\varphi]\!]^{\mathcal{M}}$. □

So if we restrict ourselves to injectively bounded frames, we get a logic that validates $\mathcal{R}(\varphi)$. This seems like a natural motivation for adopting the class of injectively bounded frames as a candidate for the class of event ordering frames for the joint logic of comparative likelihood and epistemic possibility. But it can be shown that the class of injectively bounded frames is precisely the class of Noetherian frames. To see this we'll first show that being injectively bounded guarantees being Noetherian.

Proposition 4.4. *If $\mathcal{F}$ is injectively bounded, then $\mathcal{F}$ is Noetherian.*

Proof. Suppose $\mathcal{F} = \langle W, \leq \rangle$ is injectively bounded. And suppose for reductio that there is an infinite sequence of distinct worlds:

$$X = \{w_1 \preceq w_2 \preceq w_3 \dots\}$$

Now define X' to be the set

$$X' = \{w_2 \preceq w_4 \preceq w_6 \dots\}$$

Define a function f from $W = X \cup (W \setminus X)$ to $X'' = X' \cup (W \setminus X)$ by

$$f(x) = \begin{cases} w_{2n} \text{ if } x = w_n \\ x \text{ otherwise} \end{cases}$$

Then $f : W \to X''$ is a non-decreasing injection and so $W \leq X''$. But since $X'' \neq W$, this contradicts the fact that $\mathcal{F}$ is bounded by an above lemma. □

The converse can also be shown.

Proposition 4.5. *If $\mathcal{F}$ is Noetherian, then $\mathcal{F}$ is injectively bounded.*

Proof. Let $\mathcal{F} = \langle W, \leq \rangle$ be a Neotherian frame and suppose for reductio that it is not injectively bounded. Then there is some $X \neq W$ such that $W \leq X$. That is, there is an injection $f : W \to X$ with $w \preceq f(w)$ for all $w \in W$. Note that since $X \neq W$, $W \setminus X$ is nonempty. So let $w_0 \in W \setminus X$ be arbitrary. We define a sequence

$$w_0 \preceq w_1 \preceq w_2 \ldots$$

of elements of X by setting $w_{n+1} = f(w_n)$. Since $\mathcal{F}$ is Noetherian, some element w_i of this sequence must have occurred in some earlier position in the sequence. Let n be the least number such that $w_n = w_m$ for some $m < n$. Note that since $w_n \in X$ we know that $w_m \neq w_0$. Hence $m > 0$. But then $w_n = f(w_{n-1})$ and $w_m = f(w_{m-1})$. And so if $w_n = w_m$, $w_{n-1} = w_{m-1}$ since f is injective. But since $m - 1 < n - 1$ this contradicts the assumption that n was the least such number with $w_n = w_m$ for some $m < n$. □

Thus from the perspective of event ordering semantics, the Noetherian constraint looks quite natural, since it follows from the joint assumptions that (i) the comparative confidence ordering on propositions is induced injectively from an underlying pre-order and (ii) the necessary proposition is the unique proposition at least a likely as any other. But both (i) and (ii) strike me as plausible assumptions (one for intuitive reasons and the other for more theoretical ones).

References

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